New non-Noetherian Symmetries and Multi-Hamiltonian Structures for the Toda Lattice

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Abstract.

New symmetry transformations for the n-dimensional Toda lattice are presented. Their existence allows for the construction of several first order Lagrangian structures associated to them. The multi-Hamiltonian structures are derived from Lagrangians in detail. The set of symmetries generates a Lie algebra.

Keywords: Toda Lattice, Non-Noetherian Symmetries, Multi-Hamiltonian systems, Multi-Lagrangian systems.

PACS numbers: 02.20.Sv, 02.90.+p

1. Introduction

The Toda chain [1, 2] is a well known non trivial exactly solvable model which considers a lattice where each site interacts with its nearest neighbors. This lattice has been very extensively studied [3], using different approaches in Lie algebras [4], as well as in quantum [5, 6, 7], and in relativistic systems [8, 9, 10]. Besides these studies, other generalizations have been carried out [11, 12].

In this paper, we focus in the first order approach for the Toda lattice and show its rich structures. Recently, Chavchadnize [13] has found a new symmetry transformation for the two dimensional Toda lattice. In this note, we present an extension of this symmetry transformation to n-dimensional lattices as well as four new symmetry transformations. Using these symmetries we construct new different Lagrangian structures for the n dimensional Toda model. Each one of these structures gives rise to new Hamiltonian functions and different (but equivalent) first order Euler Lagrange equations. Moreover, the new set of symmetry transformations presented here generates a Lie algebra.

2. Definitions for the Toda Lattice. First and second order formalisms.

The n dimensional Toda lattice may be described by the second order Lagrangian $L_{(2)}$ which is a function of n independent variables q^i and their time derivatives $(i, j = 1, ..., n)$

$$L_{(2)}(q^i, \dot{q}^j) = \frac{1}{2}\sum_{k=1}^{n}(\dot{q}^k)^2 - \sum_{k=1}^{n-1} e^{q^k - q^{k+1}}\,, \tag{1}$$

which gives rise to the following n second order Euler–Lagrange equations

$$\ddot{q}^k - e^{(q^{k-1}-q^k)} + e^{(q^k - q^{k+1})} = 0\,, \tag{2}$$

where $k = 1, ..., n$ and the conventions

$$\begin{aligned} e^{q^0 - q^1} &\equiv 0\,,\\ e^{q^n - q^{(n+1)}} &\equiv 0\,, \end{aligned} \tag{3}$$

have been used.

The Hamiltonian structure associated to the Lagrangian formulation (1) is defined by

$$H(q^i, p_j) = \frac{1}{2}\sum_{k=1}^{n}(p_k)^2 + \sum_{k=1}^{n-1} e^{q^k - q^{k+1}}\,, \tag{4}$$

with $i, j = 1, ..., n$ and where the momenta p_j are defined as usual by

$$p_j \equiv \frac{\partial L}{\partial \dot{q}^j} = \dot{q}^j\,. \tag{5}$$

Let us define $2n$ variables by

$$x^i = q^i\,, \tag{6a}$$
$$x^{n+j} = p_j\,. \tag{6b}$$

Thus, the Toda model Hamiltonian (4) can be written as

$$H = \frac{1}{2}\sum_{j=1}^{n}(x^{n+j})^2 + \sum_{i=1}^{n-1} e^{x^i - x^{i+1}}\,, \tag{7}$$

and the Toda model first order Lagrangian is

$$L_{(1)} = \sum_{i=1}^{n} x^{n+i}\dot{x}^i - \frac{1}{2}\sum_{i=1}^{n}(x^{n+i})^2 - \sum_{i=1}^{n-1} e^{x^i - x^{i+1}}\,. \tag{8}$$

This Lagrangian gives rise to $2n$ first order equations

$$\dot{x}^i = x^{n+i} \tag{9a}$$
$$\dot{x}^{n+j} = e^{(x^{j-1}-x^j)} - e^{(x^j - x^{j+1})} \tag{9b}$$

where $i, j = 1, ..., n$, and the conventions

$$\begin{aligned} e^{x^0 - x^1} &\equiv 0\\ e^{x^n - x^{(n+1)}} &\equiv 0 \end{aligned} \tag{10}$$

are used.

These $2n$ equations are equivalent to the previous n second order Equation (2). These first order equations can be written as $(a, b = 1, ..., 2n)$

$$\dot{x}^a = f^a(x^b, t)\,, \tag{11}$$

and

$$\begin{aligned} f^j \quad &= x^{n+j}\,, \\ f^{n+j} &= e^{(x^{j-1}-x^j)} - e^{(x^j-x^{j+1})}\,, \end{aligned} \tag{12}$$

with $j = 1, ..., n$ and where the previous conventions apply.

A symmetry transformation for a system of differential equations is defined by an infinitesimal transformation of the variables x^a

$$x'^a = x^a + \epsilon\eta^a(x^b, t)\,, \tag{13}$$

such that x'^a satisfies Equation (11) if x^a does, i.e.,

$$\dot{x}'^a = f^a(x'^b, t)\,. \tag{14}$$

Therefore, the vector $\eta^a(x^b, t)$ satisfies

$$\frac{\partial\eta^a(x^b,t)}{\partial t} + \frac{\partial\eta^a(x^b,t)}{\partial x^c} f^c - \frac{\partial f^a(x^b,t)}{\partial x^c}\eta^c = 0 \tag{15}$$

up to first order in ϵ. Note that these symmetries fulfill $(j = 1, ..., n)$

$$\eta^{n+j} = \frac{\bar{d}\eta^j}{dt} = \frac{\partial\eta^j}{\partial x^a} f^a + \frac{\partial\eta^j}{\partial t}\,. \tag{16}$$

The Equation (15) is equivalent to the so called Master Equation [14]

$$\left(\frac{\partial}{\partial t} + \underset{f}{\mathcal{L}}\right)\eta^a = 0\,, \tag{17}$$

where $\underset{f}{\mathcal{L}}$ is the Lie derivative along the vector f^a [15], which, for a vector η^a may be written as

$$\underset{f}{\mathcal{L}}\,\eta^a = \frac{\partial\eta^a}{\partial x^b} f^b - \eta^b \frac{\partial f^a}{\partial x^b}\,. \tag{18}$$

Note that $\underset{f}{\mathcal{L}}\,\eta^a = -\underset{\eta}{\mathcal{L}}\,f^a$.

3. New Symmetry Transformations for the Toda Lattice

We will exhibit five different solutions to Equation (15) for the Toda Model, with f^a given in (12). These symmetry vectors give rise to a Lie algebra where the commutation operation is defined by the Lie derivative. Further details are given below.

Chavchanidze [13] found a non trivial symmetry vector for the two dimensional Toda lattice. In fact, Chavchanidze showed that the vector $\eta_{(1)} = (\eta^1_{(1)}, \eta^2_{(1)}, \eta^3_{(1)}, \eta^4_{(1)})$ with

$$\begin{aligned}
\eta^1_{(1)} &= 2x^3 + \frac{1}{2}x^4 + \frac{t}{2}\left(\left(x^3\right)^2 + e^{(x^1-x^2)}\right) , \\
\eta^2_{(1)} &= x^4 - \frac{1}{2}x^3 + \frac{t}{2}\left(\left(x^4\right)^2 + e^{(x^1-x^2)}\right) , \\
\eta^3_{(1)} &= \frac{1}{2}\left(x^3\right)^2 - e^{(x^1-x^2)} - \frac{t}{2}\left(x^3 + x^4\right) e^{(x^1-x^2)} , \\
\eta^4_{(1)} &= \frac{1}{2}\left(x^4\right)^2 + 2e^{(x^1-x^2)} + \frac{t}{2}\left(x^3 + x^4\right) e^{(x^1-x^2)} ,
\end{aligned} \tag{19}$$

is a symmetry vector for Equation (11) with $n = 2$, i.e., the symmetry vector satisfies Equation (15).

For $n = 3$ (a slightly incorrect version of this symmetry transformation appears in [13]), it can be shown that $\eta_{(1)} = (\eta^1_{(1)}, \eta^2_{(1)}, \eta^3_{(1)}, \eta^4_{(1)}, \eta^5_{(1)}, \eta^6_{(1)})$ is a symmetry vector for Equation (11) with

$$\begin{aligned}
\eta^1_{(1)} &= 3x^4 + \frac{1}{2}x^5 + \frac{1}{2}x^6 + \frac{t}{2}\left(\left(x^4\right)^2 + e^{(x^1-x^2)}\right) , \\
\eta^2_{(1)} &= 2x^5 - \frac{1}{2}x^4 + \frac{1}{2}x^6 + \frac{t}{2}\left(\left(x^5\right)^2 + e^{(x^1-x^2)} + e^{(x^2-x^3)}\right) , \\
\eta^3_{(1)} &= x^6 - \frac{1}{2}x^4 - \frac{1}{2}x^5 + \frac{t}{2}\left(\left(x^6\right)^2 + e^{(x^2-x^3)}\right) , \\
\eta^4_{(1)} &= \frac{1}{2}\left(x^4\right)^2 - 2e^{(x^1-x^2)} - \frac{t}{2}\left(x^4 + x^5\right) e^{(x^1-x^2)} , \\
\eta^5_{(1)} &= \frac{1}{2}\left(x^5\right)^2 + 3e^{(x^1-x^2)} - e^{(x^2-x^3)} + \frac{t}{2}\left(x^4 + x^5\right) e^{(x^1-x^2)} \\
&\quad - \frac{t}{2}\left(x^5 + x^6\right) e^{(x^2-x^3)} , \\
\eta^6_{(1)} &= \frac{1}{2}\left(x^6\right)^2 + 2e^{(x^2-x^3)} + \frac{t}{2}\left(x^5 + x^6\right) e^{(x^2-x^3)} .
\end{aligned} \tag{20}$$

We have generalized Chavchanidze's result for the n dimensional Toda lattice symmetry vector $\eta_{(1)} = (\eta^j_{(1)}, \eta^{n+j}_{(1)})$ with $j = 1, \ldots.., n$ as follows

$$\begin{aligned}
\eta^j_{(1)} \; &= (n+1-j)x^{n+j} - \frac{1}{2}\sum_{k=1}^{j-1} x^{n+k} + \frac{1}{2}\sum_{k=j+1}^{n} x^{n+k} \\
&\quad + \frac{t}{2}\left((x^{n+j})^2 + e^{(x^{j-1}-x^j)} + e^{(x^j-x^{j+1})}\right) , \\
\eta^{n+j}_{(1)} &= \frac{1}{2}(x^{n+j})^2 + (n+2-j)e^{(x^{j-1}-x^j)} - (n-j)e^{(x^j-x^{j+1})} + \\
&\quad + \frac{t}{2}(x^{n+j-1} + x^{n+j})e^{(x^{j-1}-x^j)} - \frac{t}{2}(x^{n+j} + x^{n+j+1})e^{(x^j-x^{j+1})} ,
\end{aligned} \tag{21}$$

where conventions (10) must be used.

It is straightforward to prove that $\eta_{(2)} = (\eta^j_{(2)}, \eta^{n+j}_{(2)})$ with $j = 1, ..., n$ and

$$\begin{aligned} \eta^j_{(2)} &= j - \frac{t}{2} x^{n+j} \, , \\ \eta^{n+j}_{(2)} &= -\frac{1}{2} x^{n+j} - \frac{t}{2} \left(e^{(x^{j-1} - x^j)} - e^{(x^j - x^{j+1})} \right) \, , \end{aligned} \tag{22}$$

is a new symmetry vector for the n dimensional Toda Lattice.

The third symmetry vector $\eta_{(3)} = (\eta^j_{(3)}, \eta^{n+j}_{(3)})$ with $j = 1, ..., n$ is

$$\begin{aligned} \eta^j_{(3)} &= t \, , \\ \eta^{n+j}_{(3)} &= 1 \, . \end{aligned} \tag{23}$$

This symmetry transformation has an interesting feature, it is, in some sense, the inverse symmetry of $\eta_{(1)}$. In fact, we have that

$$\sigma^{(0)} = \underset{\eta_{(3)}}{\mathcal{L}} \underset{\eta_{(1)}}{\mathcal{L}} \sigma^{(0)} \, , \tag{24}$$

where $\sigma^{(0)}$ will be displayed later. The specific details will be discussed in Appendix 2.

The fourth symmetry vector $\eta_{(4)} = (\eta^j_{(4)}, \eta^{n+j}_{(4)})$ with $j = 1, ..., n$ is

$$\begin{aligned} \eta^j_{(4)} &= 1 \, , \\ \eta^{n+j}_{(4)} &= 0 \, . \end{aligned} \tag{25}$$

The last and fifth symmetry vector $\eta_{(5)} = (\eta^j_{(5)}, \eta^{n+j}_{(5)})$ is

$$\begin{aligned} \eta^j_{(5)} &= \sum_{i=1}^{n} x^i \, , \\ \eta^{n+j}_{(5)} &= \sum_{i=1}^{n} x^{n+i} \, . \end{aligned} \tag{26}$$

4. Multi-Lagrangian and multi-Hamiltonian structures

Now, we will show how the five symmetries allow us to construct new Lagrangians different from Lagrangian (8). Examples of this Multi-Lagrangian structure in two dimensions are presented.

One way of constructing new Lagrangian structures is described in Appendix 1.

4.1. Multi-Lagrangian and multi-Hamiltonian structures associated to symmetry $\eta_{(1)}$

The n dimensional Lagrangian given by (8) may be rewritten as $\hat{L}$

$$\hat{L} = \sum_{a=1}^{a=2n} \hat{l}_a \dot{x}^a + \hat{l}_0 \, , \tag{27}$$

where $\hat{l}_j = x^{n+j}$, $\hat{l}_{j+n} = 0$ and

$$\hat{l}_0 = -\frac{1}{2} \sum_{j=1}^{n} (x^{j+n})^2 - \sum_{j=1}^{n} e^{x^j - x^{j+1}} \, , \tag{28}$$

with $j = 1, \ldots\ldots, n$.

We now define a new Lagrangian L by adding a total time derivative of a function λ to Lagrangian $\hat{L}$ such that the Lagrangian one-form $l^{(0)}{}_a$, defined below, satisfies the Master Equation,

$$L = \hat{L} + \frac{d\lambda}{dt}\,, \tag{29}$$

which may be also be written as $(a = 1, ..., 2n)$

$$L = l^{(0)}{}_a(\dot{x}^a - f^a)\,. \tag{30}$$

The Lagrangian one-form $l^{(0)}{}_a$ is defined by

$$l^{(0)}{}_a = \hat{l}_a + \lambda_{,a}\ , \tag{31}$$

where the λ function must satisfy

$$\frac{\partial\lambda}{\partial x^a} f^a + \frac{\partial\lambda}{\partial t} = -\frac{1}{2}\sum_{j=1}^{n}(x^{j+n})^2 + \sum_{j=1}^{n} e^{x^j - x^{j+1}}\,. \tag{32}$$

One solution for λ satisfying Equation (32) is

$$\lambda = -\frac{t}{2}\sum_{j=1}^{n}(x^{j+n})^2 - t\sum_{j=1}^{n} e^{x^j - x^{j+1}} + \sum_{j=1}^{n}(2j-1)x^{j+n}\,, \tag{33}$$

and then using eq.(31) we have

$$\begin{aligned} l^{(0)}{}_j \quad &= x^{n+j} + t\left(e^{(x^{j-1}-x^j)} - e^{(x^j - x^{j+1})}\right)\,, \\ l^{(0)}{}_{n+j} &= -tx^{n+j} + (2j-1)\,. \end{aligned} \tag{34}$$

On the other hand, from Lagrangian (30) we get the canonical momenta

$$p^{(0)}{}_j = \frac{\partial L}{\partial \dot{x}^j} = l^{(0)}{}_j\,. \tag{35}$$

Besides, we construct the 2n-dimensional matrix $\sigma^{(0)}{}_{ab} = l^{(0)}{}_{a,b} - l^{(0)}{}_{b,a}$

$$\sigma^{(0)}{}_{ab} = \begin{pmatrix} \mathbf{0}_{N\times N} & \mathbf{1}_{N\times N} \\ -\mathbf{1}_{N\times N} & \mathbf{0}_{N\times N} \end{pmatrix}\,. \tag{36}$$

Using $l^{(0)}$ and matrix (36) we will get the Hamiltonian given by Equation (7). We can construct a Lagrangian one-form $l^{(1)}$ that satisfies the Master Equation as

$$l^{(1)}{}_a = \underset{\eta_{(1)}}{\mathcal{L}}\ l^{(0)}{}_a\,, \tag{37}$$

where $\eta_{(1)}$ is the generalized Chavchanidze's symmetry given by eq.(21). Then, the new Lagragian $L^{(1)}$ is

$$L^{(1)} = l^{(1)}{}_a\,(\dot{x}^a - f^a) = l^{(1)}{}_a\dot{x}^a + l^{(1)}_0\,, \tag{38}$$

with $l^{(1)}_0 = -l^{(1)}{}_a f^a$ $(a = 1, ..., 2n)$.

We can construct a new Hamiltonian by using the equations of motion. The Lagrangian one-form ${l^{(1)}}_a$ gives rise to a new matrix ${\sigma^{(1)}}_{ab} = {l^{(1)}}_{a,b} - {l^{(1)}}_{b,a}$. It can be proved that, when $\frac{\partial}{\partial t}{\sigma^{(1)}}_{ab} = 0$ [15] the equations of motion are

$$ {\sigma^{(1)}}_{ab}\dot{x}^b = l^{(1)}_{0\ ,a} - {l^{(1)}}_{a,0} = -\frac{\partial H^{(1)}}{\partial x^a} \,. \tag{39} $$

We could, in principle, find the Hamiltonian $H^{(1)}$ using our knowledge of ${l^{(1)}}_a$ and $l^{(1)}_0$. We can iterate this procedure to find a new Lagrangian one-form as ${l^{(2)}}_a = \underset{\eta_{(1)}}{\mathcal{L}}\, {l^{(1)}}_a$, a new matrix ${\sigma^{(2)}}_{ab} = {l^{(2)}}_{a,b} - {l^{(2)}}_{b,a}$ and a new Hamiltonian $H^{(2)}$ that give rise to the same equations of motion

$$ {\sigma^{(2)}}_{ab}\dot{x}^b = l^{(2)}_{0\ ,a} - {l^{(2)}}_{a,0} = -{H^{(2)}}_{,a} \,. \tag{40} $$

Proceeding in the same fashion, we can construct a Multi-Lagrangian system.

Let's see an example in two dimensions ($n = 2$). With two particles, the first order Toda model Lagrangian $\hat{L} = l_a\dot{x}^a + l_0$ is

$$ \hat{L} = x^3\dot{x}^1 + x^4\dot{x}^2 - \left(\frac{1}{2}(x^3)^2 + \frac{1}{2}(x^4)^2 + e^{x^1-x^2}\right) \,, \tag{41} $$

and the Euler-Lagrange equations are

$$ \begin{aligned} \dot{x}^1 &= x^3 = f^1 \,, \\ \dot{x}^2 &= x^4 = f^2 \,, \\ \dot{x}^3 &= -e^{x^1-x^2} = f^3 \,, \\ \dot{x}^4 &= e^{x^1-x^2} = f^4 \,. \end{aligned} \tag{42} $$

We can write the Lagrangian (41) as

$$ L^{(0)} = {l^{(0)}}_a(\dot{x}^a - f^a) \tag{43} $$

where

$$ \begin{aligned} l^{(0)} &= \left({l^{(0)}}_1, {l^{(0)}}_2, {l^{(0)}}_3, {l^{(0)}}_4\right) \,, \\ &= (x^3 - te^{x^1-x^2},\ x^4 + te^{x^1-x^2},\ 1 - tx^3,\ 3 - tx^4) \,. \end{aligned} \tag{44} $$

The Lagragian (43) is equivalent to (41) because they produce the same equations of motion. The momenta will be

$$ \begin{aligned} {p^{(0)}}_1 &= \frac{\partial L^{(0)}}{\partial \dot{x}^1} = {l^{(0)}}_1 \,, \\ {p^{(0)}}_2 &= \frac{\partial L^{(0)}}{\partial \dot{x}^2} = {l^{(0)}}_2 \,. \end{aligned} \tag{45} $$

We can construct the matrix

$$ {\sigma^{(0)}}_{ab} = {l^{(0)}}_{a,b} - {l^{(0)}}_{b,a} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \,, \tag{46} $$

and, if we define $l_0^{(0)} = -l^{(0)}{}_a f^a$, then the equations of motion will be

$$H^{(0)}{}_{,a} = l^{(0)}{}_{a,0} - l_0^{(0)}{}_{,a}\,, \tag{47}$$

and from here, it is possible to calculate the first Hamiltonian

$$H^{(0)} = \frac{1}{2}(x^3)^2 + \frac{1}{2}(x^4)^2 + e^{x^1-x^2}\,. \tag{48}$$

Now, we can use the Lie derivative to calculate more Lagrangians. In this way, we get

$$\begin{aligned} l^{(1)}{}_a &= \underset{\eta_{(1)}}{\mathcal{L}}\, l^{(0)}{}_a \\ &= \frac{1}{2}\Big((x^3)^2 + e^{x^1-x^2}(8 - t(x^3+x^4)),\ (x^4)^2 + e^{x^1-x^2}(-6 + t(x^3+x^4)), \\ &\quad -te^{x^1-x^2} + 6x^3 - t(x^3)^2 - x^4,\ -te^{x^1-x^2} + x^3 + 8x^4 - t(x^4)^2\Big)\,, \end{aligned} \tag{49}$$

with $\eta_{(1)}$ given in (19). Using Equation (49) we can write the second Lagrangian

$$L^{(1)} = l^{(1)}{}_a(\dot{x}^a - f^a)\,. \tag{50}$$

The Lagrange brackets matrix is

$$\sigma^{(1)}{}_{ab} = l^{(1)}{}_{a,b} - l^{(1)}{}_{b,a} = \begin{pmatrix} 0 & -e^{x^1-x^2} & x^3 & 0 \\ e^{x^1-x^2} & 0 & 0 & x^4 \\ -x^3 & 0 & 0 & -1 \\ 0 & -x^4 & 1 & 0 \end{pmatrix}, \tag{51}$$

and we can define the Strong Symmetry matrix $\Lambda^{(1)}$ by

$$\Lambda^{(1)} = \sigma^{(1)}\left(\sigma^{(0)}\right)^{-1} = \begin{pmatrix} x^3 & 0 & 0 & e^{x^1-x^2} \\ 0 & x^4 & -e^{x^1-x^2} & 0 \\ 0 & -1 & x^3 & 0 \\ 1 & 0 & 0 & x^4 \end{pmatrix}. \tag{52}$$

It is worth mentioning that the Master equation for the strong symmetry matrix Λ is equivalent to the Lax equation [16].

Using the Lagrangian (50), we can construct a new Hamiltonian for the same system. This second Hamiltonian $H^{(1)}$ is related to $H^{(0)}$ by

$$\frac{\partial H^{(1)}}{\partial x^a} = \Lambda^{(1)}{}_a{}^b\ \frac{\partial H^{(0)}}{\partial x^b}\,, \tag{53}$$

because $\sigma^{(1)}$ does not depend explicity on time.

Thus, the Hamiltonian that satisfies the Equation (39) and Equation (53), with $H^{(0)}$ given in Equation (48) and $l_0^{(1)} = -l^{(1)}{}_a f^a$ is

$$H^{(1)} = \frac{1}{3}\left(\left(x^3\right)^3 + \left(x^4\right)^3\right) + \left(x^3 + x^4\right)e^{x^1-x^2}\,, \tag{54}$$

and, if we define the Poisson Brackets $J^{(1)ab} = -\left(\sigma^{(1)ab}\right)^{-1}$ the equations of motion are

$$\dot{x}^a = J^{(1)ab}\frac{\partial H^{(1)}}{\partial x^b}\,, \tag{55}$$

and they coincide with those of the eq.(42).

We construct $l^{(2)}{}_a = \mathcal{L}_{\eta^{(1)}} l^{(1)}{}_a$

$$\begin{aligned} l^{(2)} = \frac{1}{2}\Big(& -te^{2x^1-2x^2} + (x^3)^3 - e^{x^1-x^2}\left(t(x^3)^2 + x^4(-13+tx^4) + x^3(-14+tx^4)\right), \\ & te^{2x^1-2x^2} + (x^4)^3 + e^{x^1-x^2}\left(t(x^3)^2 + x^3(-11+tx^4) + x^4(-10+tx^4)\right), \\ & -(x^3)^2(-11+tx^3) + x^3x^4 + (x^4)^2 + e^{x^1-x^2}\left(11 - t(2x^3+x^4)\right), \\ & (x^3)^2 + x^3x^4 + (x^4)^2(13-tx^4) + e^{x^1-x^2}\left(13 - t(x^3+2x^4)\right)\Big) . \end{aligned} \tag{56}$$

Thus, the Lagrangian $L^{(2)}$ may be written as

$$L^{(2)} = l^{(2)}{}_a(\dot{x}^a - f^a) , \tag{57}$$

and the Lagrange brackets matrix $\sigma^{(2)}{}_{ab} = l^{(2)}{}_{a,b} - l^{(2)}{}_{b,a}$ is

$$\sigma^{(2)}{}_{ab} = \begin{pmatrix} 0 & -\frac{3}{2}e^{x^1-x^2}\left(x^3+x^4\right) & \frac{3}{2}\left(\left(x^3\right)^2 + e^{x^1-x^2}\right) & 0 \\ \frac{3}{2}e^{x^1-x^2}\left(x^3+x^4\right) & 0 & 0 & \frac{3}{2}\left(\left(x^4\right)^2 + e^{x^1-x^2}\right) \\ -\frac{3}{2}\left(\left(x^3\right)^2 + e^{x^1-x^2}\right) & 0 & 0 & -\frac{3}{2}\left(x^3+x^4\right) \\ 0 & -\frac{3}{2}\left(\left(x^4\right)^2 + e^{x^1-x^2}\right) & \frac{3}{2}\left(x^3+x^4\right) & 0 \end{pmatrix}$$

and due to the fact

$$\frac{\partial \sigma^{(2)}{}_{ab}}{\partial t} = 0 , \tag{58}$$

we can construct a third Hamiltonian that satisfies

$$H^{(2)}{}_{,a} = l^{(2)}{}_{a,0} - l_0^{(2)}{}_{,a} = \Lambda^{(2)}{}_a{}^b \; H^{(1)}{}_{,b} , \tag{59}$$

where $l_0^{(2)} = -l^{(2)}{}_a f^a$ and $\Lambda^{(2)} = \sigma^{(2)}\left(\sigma^{(1)}\right)^{-1} = \frac{3}{2}\Lambda^{(1)}$. The Hamiltonian that satisfies Equation (59) is

$$\begin{aligned} H^{(2)} = & \frac{3}{4}e^{2x^1-2x^2} \\ & + \frac{3}{2}e^{x^1-x^2}\left(\left(x^3\right)^2 + x^3x^4 + \left(x^4\right)^2\right) + \frac{3}{8}\left(\left(x^3\right)^4 + \left(x^4\right)^4\right) , \end{aligned} \tag{60}$$

The Hamiltonian equations associated to $H^{(2)}$ and the Poisson Brackets $J^{(2)ab}$ are

$$\dot{x}^a = J^{(2)ab}\frac{\partial H^{(2)}}{\partial x^b} , \tag{61}$$

and they are identical to those of eq.(42), where $J^{(2)} = -\left(\sigma^{(2)}\right)^{-1}$.

We may proceed in the same fashion constructing Lagrangian stuctures, because the Lagrange brackets σ matrices are time independent. For example, the fourth Lagrangian for two dimensions is

$$L^{(3)} = l^{(3)}{}_a(\dot{x}^a - f^a) , \tag{62}$$

where $l^{(3)}{}_a = \underset{\eta^{(1)}}{\mathcal{L}} l^{(2)}{}_a$ is

$$\begin{aligned}
l^{(3)} = \frac{3}{4} \Big(& (x^3)^4 - 2e^{2x^1-2x^2} \left(t(x^3+x^4) - 9 \right) - \\
& -e^{x^1-x^2} \left[t(x^3)^3 + (x^3)^2(-20+tx^4) + x^3x^4(tx^4-19) + (x^4)^2(tx^4-18) \right], \\
& (x^4)^4 + 2e^{2x^1-2x^2} \left(t(x^3+x^4) - 8 \right) + \\
& +e^{x^1-x^2} \left[t(x^3)^3 + (x^3)^2(tx^4-16) + x^3x^4(tx^4-15) + (x^4)^2(tx^4-14) \right], \\
& -e^{2x^1-2x^2} t - \left((x^3)^3(tx^3-16) + (x^3)^2x^4 + x^3(x^4)^2 + (x^4)^3 \right) - \\
& e^{x^1-x^2} \left[3t(x^3)^2 + 2x^3(tx^4-16) + x^4(tx^4-15) \right], \\
& -e^{2x^1-2x^2} t + (x^3)^3 + (x^3)^2x^4 + x^3(x^4)^2 + (x^4)^3(18-tx^4) - \\
& e^{x^1-x^2} \left[t(x^3)^2 + 3x^4(tx^4-12) + x^3(2tx^4-19) \right] \Big) , \qquad (63)
\end{aligned}$$

and the fouth Hamiltonian is

$$\begin{aligned}
H^{(3)} = {} & 3\left(x^3+x^4\right) e^{2x^1-2x^2} \\
& + 3e^{x^1-x^2} \left(x^3+x^4\right) \left(\left(x^3\right)^2 + \left(x^4\right)^2 \right) \\
& + \frac{3}{5} \left(\left(x^3\right)^5 + \left(x^4\right)^5 \right) . \qquad (64)
\end{aligned}$$

The algorithm may be reiterated to get new Hamiltonians. We have presented four Lagrangians structures which show the richness of the Toda model and the power of the procedure we have adopted.

4.2. Multi-Lagrangian and multi-Hamiltonian structures associated to symmetry $\eta_{(2)}$

It is possible to repeat the same previous analysis for the symmetry η_2 given in Equation (22). We use $l^{(0)}$ given in (34), the Lagrangian of Equation (30) and Lie derivative to construct other Lagrangians. However, this symmetry does not produce a new Hamiltonian structure for the Toda lattice. Each new Hamiltonian will be identical (except for a constant) to Hamiltonian $H^{(0)}$ given by Equation (7). In fact, we can construct a Hamiltonian structures appliying Lie derivatives to Hamiltonians. In this way, it can be proved for n dimensions that using $\eta_{(2)}$ we get a $H^{(m)}$ Hamiltonian as

$$\begin{aligned}
H^{(m)} = \underset{\eta_{(2)}}{\mathcal{L}} H^{(m-1)} = \left(\underset{\eta_{(2)}}{\mathcal{L}} \right)^m H^{(0)} \\
= (-1)^m H^{(0)} . \qquad (65)
\end{aligned}$$

4.3. Multi-Lagrangian and multi-Hamiltonian structures associated to symmetry $\eta_{(3)}$

We mentioned earlier that the $\eta_{(3)}$ symmetry given by (23) is related to symmetry $\eta_{(1)}$ in the following way

$$\sigma^{(0)}{}_{ab} = \underset{\eta_{(3)}}{\mathcal{L}} \underset{\eta_{(1)}}{\mathcal{L}} \sigma^{(0)}{}_{ab} , \qquad (66)$$

where $\sigma^{(0)}{}_{ab}$ is given in (46).

Thus, if we call upward hierarchy to Lagrangians and Hamiltonians constructed with symmetry $\eta_{(1)}$, we use $\eta_{(3)}$ to construct a downward hierarchy of Lagrangians and Hamiltonians starting from any Lagrangian of the upward hierarchy.

The construction of these new Lagrangians proceeds much in the same way as it was done in the last two sections. For example, in two dimensions, we can construct a new $l'^{(1)}$ as

$$\begin{aligned} l'^{(1)}{}_a &= \underset{\eta_{(3)}}{\mathcal{L}} \; l^{(2)}{}_a \\ &= \frac{3}{2}\Big((x^3)^2 + e^{x^1-x^2}(9 - t(x^3+x^4)),\ (x^4)^2 + e^{x^1-x^2}(-7 + t(x^3+x^4)), \\ &\quad -te^{x^1-x^2} - x^3(-7+tx^3) + x^4,\ -te^{x^1-x^2} + x^3 + x^4(9 - tx^4)\Big) \end{aligned} \tag{67}$$

with $l^{(2)}$ given by Equation (56). The resulting Lagrangian is

$$L'^{(1)} = l'^{(1)}{}_a(\dot{x}^a - f^a)\,, \tag{68}$$

and the Lagrange brackets and the new Hamiltonian are

$$\sigma'^{(1)}{}_{ab} = l'^{(1)}{}_{a,b} - l'^{(1)}{}_{b,a} = 3\sigma^{(1)}{}_{ab}\,, \tag{69}$$

$$H'^{(1)} = 3H^{(1)}\,, \tag{70}$$

where $\sigma^{(1)}$ is given in (51) and $H^{(1)}$ is given in (54). Applying the Lie derivative again, we can get a new $l'^{(0)}$

$$\begin{aligned} l'^{(0)}{}_a &= \underset{\eta_{(3)}}{\mathcal{L}} \; l'^{(1)}{}_a \\ &= 3\left(x^3 - te^{x^1-x^2},\ x^4 + te^{x^1-x^2},\ 3 - tx^3,\ 5 - tx^4\right)\,, \end{aligned} \tag{71}$$

giving rise to the Lagrangian

$$L'^{(0)} = l'^{(0)}{}_a(\dot{x}^a - f^a)\,, \tag{72}$$

and $l'^{(0)}$ produces

$$\sigma'^{(0)}{}_{ab} = l'^{(0)}{}_{a,b} - l'^{(0)}{}_{b,a} = 3\sigma^{(0)}{}_{ab}\,, \tag{73}$$

$$H'^{(0)} = 3H^{(0)}\,, \tag{74}$$

where $\sigma^{(0)}$ is given by Equation (36) and $H^{(0)}$ is defined in (48). The $\eta_{(3)}$ symmetry allows us to reobtain all the multi-Lagrangian and multi-Hamiltonian structure.

At this point it is not advisable to apply the Lie derivative to $l'^{(0)}$ because it gives rise to vanishing Lagrange brackets σ. To get around this problem we use the inverse matrix to the Strong Symmetry matrix (52) and $\sigma^{(0)}$ to construct a Lagrange brackets matrix $\sigma'^{(-1)}$ as

$$\sigma'^{(-1)}{}_{ab} = \left(\Lambda^{(1)}\right)^{-1}{}_a{}^c\,\sigma^{(0)}{}_{cb}\,, \tag{75}$$

which gives

$$\sigma'^{(-1)}{}_{ab} =$$
$$\begin{pmatrix} 0 & \left(e^{x^2-x^1}x^3x^4-1\right)^{-1} & x^4\left(x^3x^4-e^{x^1-x^2}\right)^{-1} & 0 \\ -\left(e^{x^2-x^1}x^3x^4-1\right)^{-1} & 0 & 0 & x^3\left(x^3x^4-e^{x^1-x^2}\right)^{-1} \\ -x^4\left(x^3x^4-e^{x^1-x^2}\right)^{-1} & 0 & 0 & \left(x^3x^4-e^{x^1-x^2}\right)^{-1} \\ 0 & -x^3\left(x^3x^4-e^{x^1-x^2}\right)^{-1} & -\left(x^3x^4-e^{x^1-x^2}\right)^{-1} & 0 \end{pmatrix}. \tag{76}$$

Using this matrix, the equations of motion can be written as

$$\begin{aligned} \sigma'^{(-1)}{}_{ab}\dot{x}^b &= \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix} \\ &= l_0^{(-1)}{}_{,a} - l^{(-1)}{}_{a,0} = -H^{(-1)}{}_{,a}\,, \end{aligned} \tag{77}$$

where the Hamiltonian is

$$H^{(-1)} = x^3 + x^4\,, \tag{78}$$

The Hamiltonian (78) can be constructed from the momentum conservation equation. The $l_a^{(-1)}$ that satisfies (77) is

$$\begin{aligned} l^{(-1)} = &\left(\frac{x^3x^4}{x^3x^4-e^{x^1-x^2}}, \frac{x^3x^4}{x^3x^4-e^{x^1-x^2}}, \right. \\ &\left. \frac{x^4}{x^3x^4-e^{x^1-x^2}}, \frac{x^3}{-x^3x^4+e^{x^1-x^2}}\right)\,, \end{aligned} \tag{79}$$

and $l_0^{(-1)} = -l^{(-1)}{}_a f^a$.

We finally get the Lagrangian $L^{(-1)} = l^{(-1)}{}_a(\dot{x}^a - f^a)$. One keeps getting new Lagrangians by applying Lie derivatives along this symmetry in the same way as it was done before. This will produce a new downward Hamiltonian hierarchy.

4.4. Multi-Lagrangian and multi-Hamiltonian structures associated to symmetry $\eta_{(4)}$

The $\eta_{(4)}$ symmetry given in eq.(25) behaves in a different fashion. Consider $l^{(0)}$ defined by Equation (34). Its Lie derivative along $\eta_{(4)}$ vanishes, i.e.,

$$l^{(1)}{}_a = \mathop{\mathcal{L}}_{\eta_{(4)}} l^{(0)}{}_a \equiv 0\,. \tag{80}$$

This feature implies that $\eta_{(4)}$ produce Lagrangians and Hamiltonians which are identically zero. This fact can be seen in a different way. The Lie derivative of $H^{(0)}$ along $\eta_{(4)}$ vanishes, i.e.,

$$H^{(1)} = \mathop{\mathcal{L}}_{\eta_{(4)}} H^{(0)} \equiv 0\,. \tag{81}$$

4.5. Multi-Lagrangian and multi-Hamiltonian structures associated to symmetry $\eta_{(5)}$

The symmetry vector $\eta_{(5)}$ given in (26) allows us to find another Lagragian structure for the Toda lattice. For n dimensional lattices, we apply the Lie derivative to $l^{(0)}$ given in (34), as we did before, to obtain $l^{(1)}$ as

$$\begin{aligned} l^{(1)}{}_j &= 2\eta_{(5)}^{n+j} = 2\sum_{i=1}^{n} x^{n+i}\,, \\ l^{(1)}{}_{n+j} &= -2t\ \eta_{(5)}^{n+j} + n^2\,, \end{aligned} \tag{82}$$

which gives rise to the first Lagrangian and Hamiltonian structure obtained by using this symmetry

$$L^{(1)} = l^{(1)}{}_a\,(\dot{x}^a - f^a) = l^{(1)}{}_a\ \dot{x}^a - 2\left(\sum_{i=1}^{n} x^{n+i}\right)^2\,, \tag{83}$$

$$H^{(1)} = \left(\sum_{i=1}^{n} x^{n+i}\right)^2\,. \tag{84}$$

This Hamiltonian is the square of the momentum which is a constant of motion for this problem. If we apply again the Lie derivative, we get

$$\begin{aligned} l^{(2)}{}_j &= 4n\ \eta_{(5)}^{n+j}\,, \\ l^{(2)}{}_{n+j} &= -4nt\ \eta_{(5)}^{n+j} + n^3\,, \end{aligned} \tag{85}$$

where the second Lagrangian and Hamiltonian are given by

$$L^{(2)} = l^{(2)}{}_a\,(\dot{x}^a - f^a) = l^{(2)}{}_a\ \dot{x}^a - 4n\left(\sum_{i=1}^{n} x^{n+i}\right)^2\,, \tag{86}$$

$$H^{(2)} = 2n\left(\sum_{i=1}^{n} x^{n+i}\right)^2\,. \tag{87}$$

Applying the same procedure again will produce results which are similar to what we have already obtained.

5. Algebra of the new symmetries

The commutator of two vector fields A^a and B^b may be constructed by defining the operators

$$\hat{A} = A^a\frac{\partial}{\partial x^a}, \qquad \hat{B} = B^b\frac{\partial}{\partial x^b}\,, \tag{88}$$

and their commutator

$$\begin{aligned} [\hat{A},\hat{B}]\phi &= A^a\frac{\partial}{\partial x^a}\left(B^b\frac{\partial\phi}{\partial x^b}\right) - B^b\frac{\partial}{\partial x^b}\left(A^a\frac{\partial\phi}{\partial x^a}\right) \\ &= \left(A^b\frac{\partial B^a}{\partial x^b} - B^b\frac{\partial A^a}{\partial x^b}\right)\frac{\partial\phi}{\partial x^a} = \left(\mathcal{L}_A B^a\frac{\partial}{\partial x^a}\right)\phi\,. \end{aligned} \tag{89}$$

In this way, the commutator of two vector fields may be defined by

$$[A, B] = \mathcal{L}_A B = -\mathcal{L}_B A\,. \tag{90}$$

We can now compute the a component of the commutators between any two symmetry vector fields $\eta_{(m)}$ and $\eta_{(n)}$ with $m, n = 1, 2, 3, 4$

$$\begin{aligned}
[\eta_{(m)}, \eta_{(n)}]^a &= \mathcal{L}_{\eta_{(m)}} \eta^a_{(n)} \\
&= \frac{\partial \eta^a_{(n)}}{\partial x^i} \eta^i_{(m)} + \frac{\partial \eta^a_{(n)}}{\partial x^{n+i}} \eta^{n+i}_{(m)} \\
&\quad - \eta^i_{(n)} \frac{\partial \eta^a_{(m)}}{\partial x^i} - \eta^{n+i}_{(n)} \frac{\partial \eta^a_{(m)}}{\partial x^{n+i}}
\end{aligned} \tag{91}$$

with $a = 1, ..., 2n$ and $i = 1, ..., n$. After some algebra it is easy to prove that the new symmetries define a Lie algebra given by the following commutation relations (all other commutators vanish)

$$\begin{aligned}
[\eta_{(1)}, \eta_{(2)}]^a &= \frac{1}{2} \eta^a_{(1)}\,, \\
[\eta_{(2)}, \eta_{(3)}]^a &= \frac{1}{2} \eta^a_{(3)}\,, \\
[\eta_{(3)}, \eta_{(1)}]^a &= \frac{3}{2}(n+1) \eta^a_{(4)} - 2\eta^a_{(2)}\,, \\
[\eta_{(3)}, \eta_{(5)}]^a &= n\ \eta^a_{(3)}\,, \\
[\eta_{(4)}, \eta_{(5)}]^a &= n\ \eta^a_{(4)}\,, \\
[\eta_{(2)}, \eta_{(5)}]^a &= \frac{1}{2} n(n+1) \eta^a_{(4)}\,, \\
[\eta_{(1)}, \eta_{(5)}]^a &= n\ \eta^j_{(1)} \eta^a_{(4)} + n\ \eta^{j+n}_{(1)} \left(1 - \eta^a_{(4)}\right) \\
&\quad + \eta^{n+j}_{(5)} \left(2\eta^a_{(2)} - \frac{3}{2}(n+1)\eta^a_{(4)}\right)\,,
\end{aligned} \tag{92}$$

with j any number such $1 \leq j \leq n$.

6. Conclusions

We have explicitly presented five different new symmetries for the dynamics defined by the n dimensional Toda lattice and the first order Lagrangians, Hamiltonians as well as the corresponding Lagrange and Strong Symmetries associated to each of them. Moreover, we showed that the commutators of the symmetry vector fields give rise to a Lie algebra.

A multi-Lagrangian structure is obtained by taking the Lie derivative of one-form Lagrangians along each of the symmetry vector fields. These Lagrangians give rise to equivalent (although not identical) equations of motion. (They are not identical because their Lagrange bracket matrices are different). By the same token, we presented Hamiltonian structures which are different from each other but are nevertheless equivalent.

Acknowledgments

F A A is very grateful to Programa MECE Educación Superior for the Doctoral Scholarship UCH0008.

Appendix A.

In the inverse problem of the calculus of variations in first order [17], we have a Lagrangian as $L = L(q^i, \dot{q}^j, t)$ where we can define the velovity variable u^j as

$$u^j = \dot{q}^j \,,$$

and we can define a new Lagrangian $\bar{L} = L(q^i, u^j, t)$ using these variables.

We define a first order Lagrangian as [15]

$$\bar{\bar{L}} = \frac{\partial \bar{L}}{\partial u^j}\left(\dot{q}^j - u^j\right) + \bar{L}(q^i, u^j, t) \,.$$

Using this Lagrangian, the equations of motion for u^j are

$$\frac{\partial^2 \bar{L}}{\partial u^i \partial u^j}\left(\dot{q}^j - u^j\right) - \frac{\partial \bar{L}}{\partial u^i} + \frac{\partial \bar{L}}{\partial u^i} = 0 \,.$$

If the matrix $\frac{\partial^2 \bar{L}}{\partial u^i \partial u^j}$ is regular, these equations are equivalent to our definition

$$\dot{q}^j - u^j = 0 \,.$$

The equations of motion for q^j

$$\frac{\partial^2 \bar{L}}{\partial q^i \partial u^j}\left(\dot{q}^j - u^j\right) - \frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial u^i}\right) + \frac{\partial \bar{L}}{\partial q^i} = 0 \,,$$

the first term is zero because $\dot{q}^j = u^j$, and the equations of motion finally are

$$-\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{q}^i}\right) + \frac{\partial \bar{L}}{\partial q^i} = 0 \,.$$

We can say that Lagrangian at first order $\bar{\bar{L}}$ is equivalent to Lagrangian at second order L. We can write $\bar{\bar{L}}$ as $(a = 1, ..., 2n)$

$$\bar{\bar{L}} = l_a \dot{x}^a + l_0 \,, \tag{A.1}$$

where

$$x^i = q^i \,,$$
$$x^{j+n} = u^j \,,$$
$$l_i = \frac{\partial \bar{L}}{\partial u^i} \,,$$
$$l_{j+n} = 0 \,,$$
$$l_0 = -\frac{\partial \bar{L}}{\partial u^i} u^i + \bar{L} \,,$$

with $i, j = 1, ..., n$. Now, we can get the equations of motion for x^a as

$$\sigma_{ab} \dot{x}^b + \frac{\partial l_a}{\partial t} - \frac{\partial l_0}{\partial x^a} = 0 \,,$$

where $\sigma_{ab} = \frac{\partial l_a}{\partial x^b} - \frac{\partial l_b}{\partial x^a}$. If $\det \sigma \neq 0$, then there is a matrix J^{ab} such $J^{ab}\sigma_{bc} = -\delta^a_c$ and

$$\dot{x}^a = J^{ab}\left(\frac{\partial l_b}{\partial t} - \frac{\partial l_0}{\partial x^b}\right) .$$

If we demand that curl of $\frac{\partial l_b}{\partial t} - \frac{\partial l_0}{\partial x^b}$ be zero, then σ does not depend on time, and

$$\frac{\partial}{\partial x^a}\left(\frac{\partial l_b}{\partial t} - \frac{\partial l_0}{\partial x^b}\right) - \frac{\partial}{\partial x^b}\left(\frac{\partial l_a}{\partial t} - \frac{\partial l_0}{\partial x^b}\right) = 0\,,$$

and this implies

$$\frac{\partial^2 l_b}{\partial x^a \partial t} - \frac{\partial^2 l_a}{\partial x^b \partial t} = 0\,,$$

or

$$\frac{\partial \sigma_{ab}}{\partial t} = 0\,,$$

and this means

$$\frac{\partial J^{ab}}{\partial t} = 0\,.$$

If σ does not depend on time, we can always find a function such that its gradient is $\frac{\partial l_b}{\partial t} - \frac{\partial l_0}{\partial x^b}$. This is the Hamiltonian function such that

$$\frac{\partial H}{\partial x^b} = \frac{\partial l_b}{\partial t} - \frac{\partial l_0}{\partial x^b} \tag{A.2}$$

and the motion equations are now the (first order) Hamilton equations

$$\dot{x}^a = J^{ab}\frac{\partial H}{\partial x^b} = f^a\,.$$

Sometimes, l_a of Equation (A.1) does not satisfy the Master equation. We can now construct a new $\bar{l}_a$ (which satisfies the Master equation) by

$$\bar{l}_a = \mathcal{L}_\eta l_a\,,$$

such that now the Lagrangian can be written as

$$L = \bar{l}_a\left(\dot{x}^a - f^a\right)\,.$$

The new σ matrix is

$$\bar{\sigma}_{ab} = \frac{\partial \bar{l}_a}{\partial x^b} - \frac{\partial \bar{l}_b}{\partial x^a}\,.$$

This construction of $\bar{\sigma}$ is equivalent to the definition $\bar{\sigma}_{ab} = \mathcal{L}_\eta \sigma_{ab}$. Using Lie derivatives it is possible to construct many Hamiltonian functions computing many l_a and using them in conjunction with Equation (A.2).

Furthermore, we can define a new matrix Λ as

$$\Lambda_a{}^b = \bar{\sigma}_{ac}\left(\sigma^{-1}\right)^{cb}\,,$$

or, in other way, $\bar{\sigma}_{ab} = \Lambda_a{}^c\sigma_{cb}$ and $\bar{J}^{ab} = J^{ac}(\Lambda^{-1})_c{}^b$.

If we have more than one Hamiltonian, say H and $\bar{H}$, both must satisfy the equation

$$\dot{x}^a = J^{ab}\frac{\partial H}{\partial x^b} = \bar{J}^{ab}\frac{\partial \bar{H}}{\partial x^b} = J^{ac}\left(\Lambda^{-1}\right)_c{}^b\frac{\partial \bar{H}}{\partial x^b}\,,$$

then, we can find a relation between H and $\bar{H}$

$$\frac{\partial \bar{H}}{\partial x^a} = \Lambda_a{}^b \frac{\partial H}{\partial x^b} \,. \tag{A.3}$$

It is possible sometimes to construct other Hamiltonian functions using (A.3) with the help of Λ.

Appendix B.

Symmetries $\eta_{(1)}$, $\eta_{(2)}$ and $\eta_{(5)}$ may be found by solving the Master equation. The symmetry $\eta_{(4)}$ is the simplest non-trivial case of a symmetry that satisfies the Master equation. However, symmetry $\eta_{(3)}$ was found in a different fashion. We were looking for a symmetry that was the "inverse" one of $\eta_{(1)}$.

We have $\sigma^{(0)}$ given by (36) and we can define the σ' matrix like $\sigma' = \mathcal{L}_{\eta_{(1)}}\sigma^{(0)}$. Let's assume that a symmetry η' such that $\sigma^{(0)} = \mathcal{L}_{\eta'}\sigma'$ exists. In other words, the η' symmetry must fulfill

$$\sigma^{(0)} = \mathcal{L}_{\eta'}\mathcal{L}_{\eta_{(1)}}\sigma^{(0)} \,.$$

We can write this equation in an extended form as

$$\begin{aligned}
\sigma^{(0)}_{ab} = & \left[\frac{\partial^2 \sigma^{(0)}_{ab}}{\partial x^c \partial x^d} \eta^d_{(1)} \eta'^c + \left(\frac{\partial \sigma^{(0)}_{ab}}{\partial x^d} \frac{\partial \eta^d_{(1)}}{\partial x^c} + \frac{\partial \sigma^{(0)}_{db}}{\partial x^c} \frac{\partial \eta^d_{(1)}}{\partial x^a} + \frac{\partial \sigma^{(0)}_{ad}}{\partial x^c} \frac{\partial \eta^d_{(1)}}{\partial x^b} \right) \eta'^c \right. \\
& \left. + \frac{\partial \sigma^{(0)}_{cb}}{\partial x^d} \eta^d_{(1)} \frac{\partial \eta'^c}{\partial x^a} + \frac{\partial \sigma^{(0)}_{ac}}{\partial x^d} \eta^d_{(1)} \frac{\partial \eta'^c}{\partial x^b} \right] \\
& + \sigma^{(0)}_{db} \frac{\partial^2 \eta^d_{(1)}}{\partial x^c \partial x^a} \eta'^c + \sigma^{(0)}_{ad} \frac{\partial^2 \eta^d_{(1)}}{\partial x^c \partial x^b} \eta'^c + \left(\sigma^{(0)}_{cd} \frac{\partial \eta^d_{(1)}}{\partial x^b} + \sigma^{(0)}_{db} \frac{\partial \eta^d_{(1)}}{\partial x^c} \right) \frac{\partial \eta'^c}{\partial x^a} \\
& + \left(\sigma^{(0)}_{ad} \frac{\partial \eta^d_{(1)}}{\partial x^c} + \sigma^{(0)}_{dc} \frac{\partial \eta^d_{(1)}}{\partial x^a} \right) \frac{\partial \eta'^c}{\partial x^b} \,.
\end{aligned} \tag{B.1}$$

The term in square brackets vanishes in our case, because the $\sigma^{(0)}$ matrix is coordinate independent. Thus, this equation reduces to

$$\begin{aligned}
\sigma^{(0)}_{ab} = & \sigma^{(0)}_{db} \frac{\partial}{\partial x^a} \left(\frac{\partial \eta^d_{(1)}}{\partial x^c} \eta'^c \right) + \sigma^{(0)}_{ad} \frac{\partial}{\partial x^b} \left(\frac{\partial \eta^d_{(1)}}{\partial x^c} \eta'^c \right) \\
& + \sigma^{(0)}_{cd} \left(\frac{\partial \eta^d_{(1)}}{\partial x^b} \frac{\partial \eta'^c}{\partial x^a} - \frac{\partial \eta^d_{(1)}}{\partial x^a} \frac{\partial \eta'^c}{\partial x^b} \right)
\end{aligned} \tag{B.2}$$

Then, η' must satisfy both Equation (B.2) and the Master equation. It is possible to choose a symmetry such that $\partial\eta'/\partial x = 0$, i.e., a symmetry which is time dependent only. The transformation $\eta_{(3)}$ is such a symmetry for dimension n.

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